A separability theorem for dynamical systems admitting alternative Lagrangian descriptions

1987 J. Phys. A: Math. Gen. 20 3225

(http://iopscience.iop.org/0305-4470/20/11/026)

J. Phys. A: Math. Gen. **20** (1987) 3225-3236. Printed in the UK

A separability theorem for dynamical systems admitting alternative Lagrangian descriptions

C Ferrario†, G Lo Vecchio†§, G Marmo‡, G Morandi†§ and C Rubano‡

† Dipartimento di Fisica dell'Università, Ferrara, Italy
‡ Dipartimento di Fisica dell'Università and INFN, Napoli, Italy
§ GNSM, CISM, Ferrara, Italy

Received 30 July 1986

Abstract. By generalising earlier results, we prove that whenever a Lagrangian dynamical system can be associated with a (1, 1)-type tensor field which is left invariant by the dynamics, satisfies the Nijenhuis condition and is compatible with the tangent bundle structure, the system necessarily decomposes into a collection of lower-dimensional mutually non-interacting Lagrangian subsystems.

1. Introduction

In recent years, there has been a renewal of interest both in tangent bundle geometry and in Lagrangian dynamics, which has revealed (or at least emphasised, mainly for physicists) some rich geometric structures which are naturally present on the tangent bundle. One such relevant geometric structure is the so-called vertical endomorphism, a (1, 1)-type tensor field whose associated Nijenhuis tensor vanishes such that its image (in $\mathscr{X}(TM)$, the algebra of vector fields on the tangent bundle TM of a differentiable manifold M) coincides with its kernel (Grifone 1972, Klein 1983, Crampin 1983a, b). This has led to the concept of an integrable almost tangent manifold (Brickell and Clark 1974), a generalisation of the tangent bundle similar to the one leading from the cotangent bundle to a general symplectic manifold.

Another problem which has witnessed a renewal of interest is the so-called inverse problem of the calculus of variations (Helmholtz 1887, Darboux 1894, Havas 1957). It has to do with the classification of the 'inequivalent' (i.e. not differing by mere multiplication through a constant, or by the addition of a 'gauge' (a total time derivative) term) Lagrangian descriptions of a given second-order dynamics on the tangent bundle (Currie and Saletan 1966, Sarlet and Cantrijn 1978, Giandolfi *et al* 1981, Henneaux 1982, 1984, Crampin 1981, 1983b, Sarlet 1983). The ambiguities that can arise in the quantisation procedure have already been pointed out (Marmo and Saletan 1978, Balachandran *et al* 1978).

Although it can be argued that the Lagrangian description is generically unique (Henneaux 1982, Marmo and Rubano 1985, 1986), the possibility of genuinely alternative Lagrangians for the same dynamical system remains an interesting problem, mainly in cases where special symmetries are present.

An offspring of the study of alternative Lagrangians has been the possibility of associating to every pair of them (assuming at least one of the Lagrangians to be regular) a (1, 1)-type tensor field connecting the corresponding Lagrangian 2-forms

(Marmo 1982, De Filippo *et al* 1982, 1984, Crampin 1983a, b, Cariñena and Ibort 1983, Henneaux 1981, Hojman and Harleston 1981, Lutzki 1982). The tensor field thus defined is also connected with the admissible Lax representations (Lax 1968, 1975, 1976) for the dynamical system at hand (De Filippo *et al* 1983, Marmo and Rubano 1983). At the same time, it turned out that the spectral properties of the above-mentioned tensor field could be used to give conditions for the complete integrability of the dynamical system, i.e. for the existence of a sufficient number (in the sense of Liouville's theorem (Arnold 1976)) of constants of the motion pairwise in involution (Crampin *et al* 1983a, b). Such a geometric characterisation of complete integrability is an interesting result in its own right, but its interest resides also in the possibility of carrying it over to the infinite-dimensional case, i.e. of using it as a tool in the study of the integrability of non-linear field theories (De Filippo *et al* 1983).

By pursuing the study of the completely integrable case, we have recently established (Ferrario *et al* 1985) a theorem proving that, if the conditions of complete integrability given by Crampin *et al* (1983a, b) are fulfilled, then one can 'diagonalise' both the dynamical system and a class of inequivalent (or 'alternative') Lagrangians by means of a suitable point transformation. In other words, there exists a system of local coordinates in which both the dynamics (represented by a second-order vector field on *TM*) and the Lagrangians split into a sum of as many mutually independent one-dimensional dynamical (Lagrangian) systems as there are degrees of freedom in the original problem.

In the present paper, we prove a similar theorem in a broader context, and also under somewhat reduced hypotheses, namely, we will prove that the theorem also holds if the system under examination is not necessarily completely integrable. The subsystems into which the original system splits will not all be one dimensional, of course. Also, we will show that one can dispense with one of the hypotheses which played a crucial role in both the proofs of Ferrario *et al* (1985) and Crampin *et al* (1983), i.e. that the eigenvalues of the (1, 1) tensor be nowhere constant. This is a desirable feature of the proof, since the above hypothesis often fails to be fulfilled, even for simple examples.

The paper is organised as follows: in § 2 we state the problem, define the (1, 1) tensor, review its main properties and discuss the Nijenhuis condition. This section also serves to define the notation which will be used throughout the paper. Section 3 will be devoted to the proof of the theorem and to the discussion of a simple example. In § 4, we discuss the results and draw some final conclusions.

2. (1, 1)-type fields of the Nijenhuis type associated with Lagrangian dynamics

Let Q be a smooth manifold, TQ its tangent bundle, $\pi: TQ \to Q$ the canonical projection. Local coordinates in a tangent bundle atlas for TQ will be denoted by (q^i, u^i), $i = 1, \ldots, n = \dim Q$, the q^i being local coordinates for the base manifold Q. Among the (intrinsic) geometric objects which can be defined on TQ, we will mainly need the following.

(i) The *field of dilations* (or Liouville field) along the fibres. It is the vertical field $\Delta \in \mathscr{X}(TQ)$ whose local expression is

$$\Delta = u^i \frac{\partial}{\partial u^i}. \tag{2.1}$$

(ii) The *vertical endomorphism* defined as the (1, 1)-type tensor field $S \in \mathcal{T}_1^1(TQ)$ with local expression:

$$S = \mathrm{d}q^i \otimes \frac{\partial}{\partial u^i}. \tag{2.2}$$

For the sake of brevity, we give here only local expressions, referring to the literature for a discussion of the properties of the above objects in intrinsic, coordinate-free, terms. It is well known that S defines the endomorphisms

$$\begin{aligned} &\hat{S} : \mathscr{X}(TQ) \to \mathscr{X}(TQ) \\ &\langle \hat{S}X | \theta \rangle =: S(X, \theta) \qquad \forall X \in \mathscr{X}(TQ), \theta \in \mathscr{X}^*(TQ) \end{aligned} \tag{2.3}$$

and the dual endomorphism:

$$\begin{aligned} &\check{S} : \mathscr{X}^*(TQ) \to \mathscr{X}^*(TQ) \\ &\langle X | \check{S}\theta \rangle =: \langle \hat{S}X | \theta \rangle \end{aligned} \tag{2.4}$$

where $\langle \, | \, \rangle$ denotes the usual pairing between vectors and 1-forms. From the definition, it follows that

$$S^2 = 0 \tag{2.5}$$

and

$$\ker \hat{S} = \operatorname{Im} \hat{S} = \mathscr{X}^{\mathrm{v}}(TQ) \tag{2.6}$$

$\mathscr{X}^{\mathrm{v}}$ being the subset of $\mathscr{X}(TQ)$ composed of the vertical vector fields (i.e.: $X \in \mathscr{X}^{\mathrm{v}}$ iff $T\pi \cdot X = 0$).

A vector field $\Gamma \in \mathscr{X}(TQ)$ is *second order* iff

$$\hat{S}\Gamma = \Delta. \tag{2.7}$$

Hence, the local expression of a second-order field will be

$$\Gamma = u^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial u^i} \qquad f^i \in \mathscr{F}(TQ). \tag{2.8}$$

For us, and from now on, *a dynamical system* will be a second-order field Γ on TQ. A function $\mathscr{L} \in \mathscr{F}(TQ)$ is an admissible Lagrangian function for Γ iff

$$L_\Gamma \theta_{\mathscr{L}} - \mathrm{d}\mathscr{L} = 0 \tag{2.9}$$

where

$$\theta_{\mathscr{L}} =: \check{S} \, \mathrm{d}\mathscr{L} \tag{2.10}$$

is the Cartan 1-form associated with $\mathscr{L}$. A function $\mathscr{L}$ will be called a *regular* Lagrangian iff the associated 2-form:

$$\Omega_{\mathscr{L}} =: -\mathrm{d}\theta_{\mathscr{L}} \tag{2.11}$$

is non-degenerate, hence a symplectic form on TQ. In this case, there exists a unique dynamical system Γ satisfying (2.9). It will be called a Lagrangian dynamical system. As is well known, this will happen iff the Hessian matrix

$$\mathscr{H} = \| H_{ij} \| \qquad H_{ij} =: \partial^2 \mathscr{L} / \partial u^i \, \partial u^j \tag{2.12}$$

is nowhere singular.

The relationship between a Lagrangian 2-form $\Omega_{\mathscr{L}}$ and the vertical endomorphism is expressed by the identity

$$\Omega_{\mathscr{L}}(\hat{S}X, Y)+\Omega_{\mathscr{L}}(X, \hat{S}Y)=0 \qquad \forall X, Y \in \mathscr{X}(TQ). \tag{2.13}$$

(For a proof, see, e.g., Crampin (1981).)

In view of (2.5), (2.13) implies

$$\Omega_{\mathscr{L}}(\hat{S}X, \hat{S}Y)=0 \qquad \forall X, Y \tag{2.14}$$

i.e. that the vertical subspaces at every point are Lagrangian subspaces for $\Omega_{\mathscr{L}}$. Also, (2.9) implies

$$L_\Gamma\Omega_{\mathscr{L}}=0. \tag{2.15}$$

We recall that, in turn, if a closed 2-form Ω satisfies (2.14), and $L_\Gamma\Omega=0$ for some second-order field Γ, then a function $\mathscr{L}$ can be found such that $\Omega=\Omega_{\mathscr{L}}$, at least locally (Balachandran *et al* 1980). Finally, using Cartan's identity, one easily obtains the Hamiltonian version of (2.9), i.e.

$$i_\Gamma\Omega_{\mathscr{L}}=\mathrm{d}E_{\mathscr{L}} \qquad E_{\mathscr{L}}=:(L_\Delta-1)\mathscr{L}. \tag{2.16}$$

We now assume that the inverse problem for the given dynamical system Γ has at least two inequivalent admissible (in the sense of § 1) Lagrangians, $\mathscr{L}, \mathscr{L}' \in \mathscr{F}(TQ)$, and that at least one of the two Lagrangians, $\mathscr{L}$, say, is a regular one.

Turning to the corresponding Lagrangian 2-forms $\Omega_{\mathscr{L}}$ and $\Omega_{\mathscr{L}'}$, $\Omega_{\mathscr{L}}$ (at least) will be a symplectic form. We can then associate to the pair of inequivalent Lagrangians $\mathscr{L}, \mathscr{L}'$, a (1, 1)-type tensor field $T \in \mathscr{T}^1_1(TQ)$, uniquely defined by the equation (Marmo and Rubano 1983, Crampin 1983b)

$$\Omega_{\mathscr{L}}(\hat{T}X, Y)=:\Omega_{\mathscr{L}'}(X, Y) \qquad \forall X, Y \in \mathscr{X}(TQ). \tag{2.17}$$

Equation (2.15) (which holds for both $\Omega_{\mathscr{L}}$ and $\Omega_{\mathscr{L}'}$) together with the definition (2.17) implies that T is invariant under the action of the dynamical vector field, i.e.

$$L_\Gamma T=0. \tag{2.18}$$

Again, (2.17) implies

$$\Omega_{\mathscr{L}}(\hat{T}X, Y)=\Omega_{\mathscr{L}}(X, \hat{T}Y) \qquad \forall X, Y \tag{2.19}$$

and, together with (2.13) (Crampin 1983a),

$$T\cdot S=S\cdot T. \tag{2.20}$$

Equation (2.20) is a compatibility condition between T and the tangent bundle structure of TQ.

Conversely, if a (1, 1)-type tensor field T is given which satisfies (2.18)-(2.20), then, defining Ω' via the converse of (2.17), i.e.

$$\Omega'(X, Y)=:\Omega_{\mathscr{L}}(TX, Y) \tag{2.21}$$

Ω' will be skew-symmetric (hence a 2-form) and will satisfy both (2.14) and (2.15). It will then be a Lagrangian 2-form (i.e. T will generate a new (possibly degenerate) Lagrangian) iff it is also closed. In what follows, we will need only properties (2.18)-(2.20) of T (plus some additional assumptions which will shortly be listed below). Therefore, the existence of the second Lagrangian $\mathscr{L}'$ is not, strictly speaking, a crucial assumption. Rather, it serves mainly the purpose of exhibiting a sufficient condition under which a mixed tensor field exists with the required properties.

As $\hat{T}(\check{T})$ is an endomorphism of $\mathscr{X}(TQ)$ ($\mathscr{X}^*(TQ)$), one can pose an eigenvalue problem for $\hat{T}(\check{T})$, at every point $m \in TQ$, in $T_m(TQ)$ ($T^*_m(TQ)$), and the eigenvalues will turn out to be smooth functions on TQ. It follows from (2.18) that (i) the eigenvalues are all constants of the motion for Γ, and from (2.20) that (ii) the degeneracy of each eigenvalue is *even* (i.e. at least double).

The *Nijenhuis tensor* (Frölicher and Nijenhuis 1956) associated with T is the (1, 2)-type tensor field $N_T \in \mathscr{T}^1_2(TQ)$ defined by

$$N_T(X, Y) = [\hat{T}X, \hat{T}Y] + \hat{T}^2[X, Y] - \hat{T}[\hat{T}X, Y] + \hat{T}[\hat{T}Y, X]. \tag{2.22}$$

We now make the following assumptions on T.

(Ai) $\hat{T}$ (resp $\check{T}$) is *diagonalisable.* By this we mean that, if λ_i, $i = 1, \ldots, r \leqslant n$ are the distinct eigenvalues of $\hat{T}$ and $\mathscr{D}_i(m)$, $m \in TQ$, the corresponding eigenspaces, then

$$\bigoplus_{i=1}^{r} \mathscr{D}_i(m) = T_m(TQ) \qquad \forall m \tag{2.23}$$

(that $r \leqslant n$ follows from (2.20)).

(Aii) The eigenvalues of $\hat{T}$ (resp $\check{T}$) have *constant degeneracy* throughout TQ. Hence, if $\dim \mathscr{D}_i(m) = 2k_i$, k_i will be independent of m and $\Sigma_{i=1}^{r} k_i = n$.

(Aiii) The *Nijenhuis condition*:

$$N_T = 0 \tag{2.24}$$

holds.

The eigenspaces of any eigenvalue of $\hat{T}$, by the assumption (Aii) of constant dimension, define a distribution on TQ. One of the main consequences of the Nijenhuis condition (2.24) is (Marmo 1982) that every such distribution is involutive, and hence, by Frobenius' theorem, integrable. Every mixed tensor field T satisfying the above assumptions then yields $r \leqslant n$ independent foliations of TQ, the dimension of each one of them being equal to the degeneracy of the corresponding eigenvalue.

The leaf of the ith foliation through any point $m \in TQ$ will be called the 'ith eigenmanifold' through m.

3. The main theorem

Having established in § 2 some general facts concerning mixed tensor fields associated with Lagrangian dynamics, we now turn to the proof of the separability theorem. In the notation (and with the assumptions) of § 2, let $\mathscr{D}_i$ be the ith eigendistribution associated with the mixed tensor field T, i.e.

$$T\mathscr{D}_i(m) = \lambda_i \mathscr{D}_i(m). \tag{3.1}$$

The assumed invariance of T WRT the action of the dynamical vector field Γ entails

$$[\Gamma, \mathscr{D}_i] \subseteq \mathscr{D}_i \qquad i = 1, \ldots, r. \tag{3.2}$$

Hence (Marmo *et al* 1985), Γ will be projectable WRT all the r foliations associated with the distinct eigenvalues of T. It follows that Γ will split into the sum of r independent vector fields:

$$\Gamma(m) = \sum_{i=1}^{r} \Gamma_{(i)}(m). \tag{3.3}$$

At every point $m \in TQ$, $\Gamma_{(i)}$ will be tangential to the ith eigenmanifold through m and will be a function only of the coordinates along the ith eigenmanifold. To be more specific, if $\{\xi^{i,s}\} i = 1, \ldots, r$, $s = 1, \ldots, 2k_i$ is a system of (collective) coordinates simultaneously adapted to the r foliations in a neighbourhood of any $m \in TQ$, with the $\xi^{i,s}$, for fixed i, being coordinates for the leaves of the ith foliation, then

$$\Gamma_{(i)} = \sum_{s=1}^{2k_i} A^s_{(i)} \frac{\partial}{\partial \xi^{i,s}} \qquad A^s_{(i)} = A^s_{(i)}(\xi^{i,s}) \qquad (i \text{ fixed}). \tag{3.4}$$

A similar splitting can be easily proved to hold for Ω, Ω being (any) one of the 2-forms generated by the alternative Lagrangians for Γ. Indeed, by (2.19)

$$\Omega(X, Y) = 0 \tag{3.5}$$

whenever X, Y belong to eigenspaces corresponding to different eigenvalues of T. It follows that

$$\Omega = \sum_{i=1}^{r} \Omega^{(i)} \tag{3.6}$$

and, in the collective coordinates introduced above

$$\Omega^{(i)} = \tfrac{1}{2} \Omega^{(i)}_{jk} \, d\xi^{i,j} \wedge d\xi^{i,k} \tag{3.7}$$

(sum over j, k, with i fixed). If Ω is non-degenerate, each one of the $\Omega^{(i)}$ will be separately non-degenerate. Moreover, the closure of Ω implies the *separate* closure of $\Omega^{(i)}$, $\forall i$. Indeed, we have from (3.7) $d\Omega = \Sigma_i \, d\Omega^{(i)}$, and

$$d\Omega^{(i)} = \frac{1}{3!} \Omega^{(i)}_{[jk,s]} \, d\xi^{i,s} \wedge d\xi^{i,j} \wedge d\xi^{i,k} + \frac{1}{2} \sum_{j \neq i} \frac{\partial \Omega^{(i)}_{rk}}{\partial \xi^{j,s}} \, d\xi^{j,s} \wedge d\xi^{i,r} \wedge d\xi^{i,k} \tag{3.8}$$

where

$$\Omega^{(i)}_{jk,s} = \partial \Omega^{(i)}_{jk} / \partial \xi^{i,s} \tag{3.9}$$

and $[\ldots]$ stands, as usual, for antisymmetrisation of indices. Clearly, there is no possible cancellation between $d\Omega^{(i)}$ and any $d\Omega^{(j)}$, $j \neq i$, so the $d\Omega^{(i)}$ must vanish separately. Moreover, vanishing of (3.8) also implies $\partial \Omega^{(i)}_{rk} / \partial \xi^{j,s} = 0$, $\forall j \neq i$, i.e. the coefficients of $\Omega^{(i)}$ can only depend on the coordinates along the ith eigenmanifold, as is the case for Γ.

With these preliminary results in mind, let us consider the ith eigendistribution $\mathscr{D}_i$ and the associated foliation of TQ. Let $VT_m(TQ)$ denote the vertical subspace of $T_m(TQ)$ and define

$$A_i(m) = \mathscr{D}_i(m) \cap VT_m(TQ). \tag{3.10}$$

We can in many ways find another subspace of $\mathscr{D}_i(m)$ to supplement A_i, i.e. a subspace $R_i(m)$ such that

$$\mathscr{D}_i(m) = A_i(m) \otimes R_i(m) \qquad R_i(m) \cap A_i(m) = \{\mathbf{0}\}. \tag{3.11}$$

We then have the following.

Lemma. If $\dim \mathscr{D}_i = 2k_i$, and the 2-form Ω is non-degenerate, then $\dim A_i(m) = k_i$.

Indeed, if S is the vertical endomorphism, $\hat{S}R_i(m) \subseteq A_i(m)$, hence $\dim(\hat{S}R_i(m)) \leq \dim A_i(m)$. Let us prove that $\dim \hat{S}R_i = \dim R_i$. If $X, Y \in R_i$ and are linearly independent, then $\hat{S}X$, $\hat{S}Y$ are also linearly independent. For, if there were two constants μ, ν such that $\mu\hat{S}X + \nu\hat{S}Y = 0$, this would imply

$$\hat{S}(\mu X + \nu Y) = 0 \Rightarrow \mu X + \nu Y \in A_i(m)$$

which would violate (3.11) (no non-zero vector of R_i can be vertical). So $\dim \hat{S}R_i = \dim R_i$ and

$$\dim A_i \geq \dim R_i.$$

As $\dim A_i + \dim R_i = 2k_i$, this implies $\dim A_i \geq k_i$.

Assume now Ω to be non-degenerate. Then, $\Omega^{(i)}$ will be a non-degenerate 2-form on each eigenmanifold belonging to the eigenvalue λ_i. A_i, being composed of vertical fields, will be an isotropic subspace for $\Omega^{(i)}$. As the latter is non-degenerate, the dimension of A_i cannot exceed k_i and this achieves the proof of the lemma.

Now let $X_1, \ldots, X_{k_i}$ be a basis for A_i. Via the symplectic form Ω, we can associate to each X_j the 1-form

$$\theta_j \overset{\text{def}}{=} i_{X_j}\Omega. \tag{3.12}$$

Note that, as $X_j \in \mathscr{D}_i$, $i_{X_j}\Omega \equiv i_{X_j}\Omega^{(i)}$. Also, using (2.13), we obtain

$$i_Z\check{S}\theta_j = i_{\hat{S}Z}\theta_j = \Omega(X_j, \hat{S}Z) = -\Omega(\hat{S}X_j, Z) = 0 \qquad \forall Z \in \mathscr{X}(TQ) \tag{3.13}$$

because X_j is a vertical field. Hence

$$\check{S}\theta_j = 0 \tag{3.14}$$

i.e. θ_j is a 'basic' 1-form. In local bundle coordinates, the form of θ_j will then be

$$\theta_j = \theta_{j,k}\,\mathrm{d}q^k \qquad \theta_{j,k} \in \mathscr{F}(TQ). \tag{3.15}$$

Also, using (3.5), it is easy to prove that the θ_j are eigenforms corresponding to the eigenvalue λ_i. Indeed

$$i_Z\check{T}\theta_j = i_{\hat{T}Z}\theta_j = \Omega(X_j, \hat{T}Z) = \Omega(\hat{T}X_j, Z) = \lambda_i\Omega(X_j, Z) = \lambda_i i_Z\theta_j \qquad \forall Z \in \mathscr{X}(TQ).$$

Hence

$$\check{T}\theta_j = \lambda_i\theta_j. \tag{3.16}$$

The invariance of T WRT the dynamics Γ implies that its eigenvalues are constants of the motion for Γ. Then taking the Lie derivative of (3.16) WRT Γ, we easily find that $L_\Gamma\theta_j$ is also an eigenform of T relative to the eigenvalue λ_i. Therefore, as the Lie derivative of a non-zero basic 1-form WRT a second-order field cannot be zero, the θ_j, together with the $L_\Gamma\,\theta_j$, will span the $2k_i$-dimensional space of eigenforms relative to the eigenvalue λ_i.

From the already mentioned fact that $i_{X_j}\Omega \equiv i_{X_j}\Omega^{(i)}$, it follows that θ_j can be considered as a 1-form on the ith eigenmanifold, and that the set of 1-form

$$\Theta = \{\theta_1, \ldots, \theta_{k_i}\} \tag{3.17}$$

can be considered as a codistribution on the same $2k_i$-dimensional manifold. From (3.12) and the fact that Ω is a Lagrangian 2-form, it follows that $A_i \subseteq \ker\Theta$. On the other hand, as $\dim A_i = k_i = \dim\ker\Theta$, A_i coincides with $\ker\Theta$. As the Lie bracket of any two vertical fields is again vertical, and from the fact that $\mathscr{D}_i$ is involutive, it follows that A_i *is involutive.* Hence, the codistribution (3.17) is integrable, i.e. there exists a matrix $\|A^{ij}\|$ of integrating factors:

$$A^{rj}\theta_j = \mathrm{d}f^r \qquad A^{ij}, f^r \in \mathscr{F}(TQ). \tag{3.18}$$

Note that, if $Z \in \mathscr{D}_k$, $k \neq i$, $i_Z\theta_j = 0$ entails

$$i_Z\,\mathrm{d}f^r = L_Zf^r = 0. \tag{3.19}$$

Hence, again the f^r depend only on the coordinate along the ith eigenmanifold. Moreover, as the θ_j are basic forms, the function f^r can depend only on the q. Let us call $f^r_{(i)}$, $r=1,\ldots,k_i$, the functions defined by the integrability condition (3.18). The set of functions

$$F: Q \to \mathbb{R}^n \qquad F:(q^1,\ldots,q^n) \to \{f^s_{(i)}(q)\} \qquad i=1,\ldots,r;\ s=1,\ldots,k_i \tag{3.20}$$

defines locally a coordinate change in the base manifold. The lifting of (3.20) to TQ

$$(q,\dot{q}) \to (F,\dot{F}) \qquad \dot{F}=\{\dot{f}^s_{(i)}\} \qquad \dot{f}^s_{(i)} \overset{\text{def}}{=} L_\Gamma f^s_{(i)} \tag{3.21}$$

is a Newtonian transformation. Both the $\mathrm{d}f^r_{(i)}$ and (taking Lie derivatives WRT Γ) the $\mathrm{d}\dot{f}^r_{(i)}$ are, for fixed i, a basis of 1-form on the ith eigenmanifold. By duality, the $(\partial/\partial f)^r_{(i)}$ and $(\partial/\partial \dot{f})^r_{(i)}$ will yield a local basis of vector fields.

We have in this way achieved a generalisation of the results of Ferrario *et al* (1985) to the case in which the degeneracy of the eigenvalues of T is not the minimum one (i.e. double), and hence complete integrability of the dynamics is not granted. The proof that the dynamical vector field Γ and (apart from irrelevant 'gauge' terms) all the T-equivalent Lagrangians split into sums of independent terms goes through exactly as in the above reference. We can then summarise the results of this section in the following.

Theorem. Let $\Gamma \in \mathscr{X}(TQ)$ be a second-order vector field admitting a Lagrangian description with a regular Lagrangian $\mathscr{L} \in \mathscr{F}(TQ)$. Let T be a (1, 1)-type tensor field satisfying (2.18)-(2.20). Such a tensor will exist if Γ admits alternative (inequivalent) Lagrangian descriptions. Let T satisfy (Ai)-(Aiii). Then one can find a Newtonian transformation (the canonical lifting to TQ of a coordinate transformation on the base manifold) leading to a coordinate system simultaneously adapted to all the foliations of TQ associated with the distinct eigenvalues of T such that:

(i) through every point $m \in TQ$, the eigenmanifold associated with any one of the eigenvalues of T is the tangent bundle of a (base) manifold whose dimension equals half the degeneracy of the eigenvalue,

(ii) both Γ, $\mathscr{L}$ and any of the T-equivalent Lagrangians split into the sum of independent second-order vector fields and ('modulo' irrelevant gauge terms) Lagrangians, one for each eigenmanifold. The 'components' of Γ and of the Lagrangians in each eigenmanifold depend solely on the coordinates on the eigenmanifold itself.

Whenever the assumptions of the theorem are fulfilled, the dynamical problem then splits directly into the sum of r independent, lower-dimensional, Lagrangian dynamical problems. In some cases, this can be of great help in integrating the dynamics. In the case of minimum degeneracy ($k_i=1$, $\forall i$, and hence $r=n$), we are of course back to the completely integrable case considered in Ferrario *et al* (1985).

As an application of the theorem, we will now briefly discuss a non-trivial completely integrable system, namely the Toda molecule (Thirring 1978). In this case, $Q=\mathbb{R}^3$ and, in cartesian coordinates, the system can be described by the regular (and in fact (Marmo *et al* 1985) hyperregular) Lagrangian:

$$\begin{aligned} &\mathscr{L} = K - V \\ &K = \tfrac{1}{2}\delta_{ij}u^i u^j \qquad V = \exp(q^1-q^2)+\exp(q^2-q^3)+\exp(q^3-q^1) \end{aligned} \tag{3.22}$$

and of course

$$\Gamma = u^i \frac{\partial}{\partial q^i} - \delta^{ij} \frac{\partial V}{\partial q^j} \frac{\partial}{\partial u^i}. \tag{3.23}$$

It has already been proved in the literature (Antonini *et al* 1985) that the requirement that the conditions expressed by (2.18)-(2.20) be fulfilled entails, in a coordinate system in which the 'forces' (i.e. the components of the vertical part of Γ) are velocity independent, that T be of the form

$$T = M_i^j \left(dq^i \otimes \frac{\partial}{\partial q^j} + du^i \otimes \frac{\partial}{\partial u^j} \right) \tag{3.24}$$

with $M_i^j = M_j^i$ and $L_\Gamma M_i^j = 0 \; \forall i, j$. Indeed, the fact that the same matrix $M = \|M_i^j\|$ represents the action of T on the horizontal and vertical subspaces ensures the compatibility of T with the tangent-bundle structure, (2.20), while the symmetry of M ensures the fulfillment of the condition expressed by (2.19).

Due to the structure of Γ (and of the Lagrangian $\mathscr{L}$), the centre-of-mass motion can be, so to speak, split off from the dynamics, for example with the aid of the Newtonian transformation relative to the centre-of-mass variables given by

$$Q^i = \lambda_j^i q^j \qquad U^i = \lambda_j^i u^j \tag{3.25}$$

where

$$\Lambda = \|\lambda_j^i\| = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{vmatrix}. \tag{3.26}$$

We therefore expect the relevant (1, 1)-type tensor T to have a doubly (i.e. minimum) degenerate eigenvalue, corresponding to the centre-of-mass motion, and a fourfold degenerate one, corresponding to the (non-trivial) dynamics in the relative coordinates. Any richer structure of T has not to be expected, as it would correspond to a complete splitting of the dynamics of the Toda molecule into three independent one-dimensional subdynamics.

With this in mind, the most general form of the matrix M can be parametrised in the form

$$M = \begin{vmatrix} A & B & B \\ B & A & B \\ B & B & A \end{vmatrix} \tag{3.27}$$

where $A, B \in \mathscr{F}(TQ)$ and $L_\Gamma A = L_\Gamma B = 0$. A matrix of the form (3.27) has the following eigenvalues:

$$h = A - B \text{ (doubly degenerate)} \qquad k = A + 2B \text{ (non-degenerate)}. \tag{3.28}$$

The tensor field T will then have the required structure. Besides being diagonalisable, it will be even invertible, provided $h \cdot k \neq 0$. As a simple check, it is easy to prove, e.g., that

$$\Lambda M \Lambda^{-1} =: \tilde{M} = \begin{vmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & k \end{vmatrix} \tag{3.29}$$

i.e. that the transformation (3.25) actually diagonalises M (and hence T). As (Ai) and (Aii) are satisfied by T, we now investigate the Nijenhuis condition. After some algebra, it can be shown that (2.24) is equivalent, in the present case, to the following set of partial differential equations:

$$M_j^{\ h}\frac{\partial M_i^{\ k}}{\partial u^h}-\frac{\partial M_i^{\ h}}{\partial u^j}M_h^{\ k}=0 \tag{3.30a}$$

and

$$M_j^{\ h}\frac{\partial M_i^{\ k}}{\partial q^h}-\frac{\partial M_i^{\ h}}{\partial q^j}M_h^{\ k}=0. \tag{3.30b}$$

The meaning of (3.30) is best seen in a basis (see above) in which T (and hence M) is diagonal. If then $M_i^j = M_i\,\delta_i^j$, $M_1 = M_2 = h$, $M_3 = k$, it can be easily shown that (3.30) yield

$$\frac{\partial k}{\partial U^i}=\frac{\partial k}{\partial Q^i}=0 \qquad i=1,2 \qquad \frac{\partial h}{\partial U^3}=\frac{\partial h}{\partial Q^3}=0. \tag{3.31}$$

In other words

$$h=h(Q^1, Q^2; U^1, U^2) \qquad k=k(Q^3; U^3). \tag{3.32}$$

As both h and k are constants of the motion, and as the only constants of the motion which depend solely on the centre-of-mass coordinates are the functions of U^3, we obtain, as a first result,

$$k=k(U^3) \tag{3.33}$$

U^3 being the centre-of-mass velocity. Therefore, any tensor field of the form (3.24), with the matrix M given by (3.27), and with the condition (3.33), will achieve the separation of the original dynamical system into two independent subsystems.

As a final comment, let us remark that, up to now, the only condition on the second eigenvalue, h, is that it be a constant of the motion of the (reduced) problem (cf (3.32)). If we require the 2-form Ω' generated via (2.21) to be a Lagrangian 2-form, we have to require it to be closed, as discussed in § 2. It is then again easy to prove that the closure condition on Ω' is equivalent to the following equations:

$$\frac{\partial h}{\partial Q^2}-\frac{\partial h}{\partial Q^1}=\frac{\partial h}{\partial U^2}-\frac{\partial h}{\partial U^1}=0 \tag{3.34}$$

which, together with (3.32), entails

$$h=h(Q^1+Q^2; U^1+U^2) \tag{3.35}$$

i.e. that h should be a constant of the motion depending only on the centre-of-mass coordinates of the *reduced* problem. As, again, the existence of such a non-trivial constant of the motion would imply complete reducibility of the full problem, we conclude that the only possibility is

$$h=\text{constant}. \tag{3.36}$$

We have in this way essentially recovered, along a different line of thought, an already established result (Antonini *et al* 1985), i.e. that the most general alternative Lagrangians for the Toda molecule are generated by 'fouling' (Currie and Saletan 1966) the one-dimensional part (centre-of-mass motion), and by multiplying the remaining (relative motion) part by a constant, i.e. by combining trivial means of separately generating alternative Lagrangians to obtain non-trivially equivalent ones.

The same situation will emerge for n-body systems interacting with gravitational forces: the splitting of T will correspond to replacing original coordinates with 'centre-of-mass' and 'internal' coordinates.

4. Discussion and conclusions

The main hypotheses of the theorem proved in § 3 are that the mixed tensor field T is invariant under the dynamics, that it satisfies the Nijenhuis condition and that it be compatible with the tangent bundle structure. Although in the example discussed at the end of § 3 the Nijenhuis condition may appear to be somewhat redundant as far as the goal of splitting the dynamics is concerned, the condition seems to play a crucial role in the proof of the theorem, as it is essential to prove the integrability of the eigendistributions of T. So, it is difficult to believe that the Nijenhuis condition can be relaxed, as it plays such a relevant geometrical role.

In constructing the proof of § 3 we have instead been able to get rid of another rather stringent condition, which was essential in the earlier version of the theorem (Ferrario *et al* 1985), namely the assumptions that the eigenvalues of T be nowhere constant:

$$\mathrm{d}\lambda_i|_m \neq 0 \qquad \forall m \in TQ, \forall i.$$

This was a technical assumption, with no geometrical counterpart, and, as again the example of § 3 shows, it could become a too stringent one even in simple and well tractable cases. Also, the earlier proof just mentioned was restricted to the completely integrable case (minimum degeneracy of the eigenvalues of T), another condition we have been able to relax in the present paper.

The main result of the paper is, therefore, that, for Lagrangian dynamical systems, the existence of a mixed tensor field invariant under the dynamics, satisfying the Nijenhuis condition and the condition of compatibility with the tangent bundle structure, forces the system to split into a collection of lower-dimensional non-interacting subsystems.

Such a splitting occurs in the Hamiltonian case as well (De Filippo *et al* 1984), i.e. when the dynamical system is given on the cotangent bundle (and in Hamiltonian form). However, in that case, the coordinate system in which the splitting occurs arises in general from a non-fibre-preserving transformation, i.e. the original coordinates and moments are 'mixed up' in the new system. In other words, in the Hamiltonian case one can relax in a more natural way the condition of compatibility with the fibre bundle structure, thus allowing for more interesting situations to occur. For instance, in the two-dimensional central force problem, a suitable mixed tensor field can be found only if we relax the compatibility condition.

The theorem proved in the present paper represents, therefore, a sort of no-interaction theorem for Lagrangian dynamical systems in the sense that the fulfillment of the Nijenhuis condition *and* the simultaneous compatibility with the tangent bundle structure are possible for a $(1, 1)$-type tensor field preserved by a second-order Lagrangian dynamical vector field iff the dynamics itself is built up from lower-dimensional and genuinely non-interacting subdynamics.

References

Antonini P, Marmo G and Rubano C 1985 *Nuovo Cimento* B **86** 17
Arnold V I 1976 *Méthodes Mathématiques de la Mécanique Classique* (Moscow: Mir)
Balachandran A P, Govindarajan J R and Vijayalakshmi B 1978 *Phys. Rev.* D **18** 1950
Balachandran A P, Marmo G, Skagerstam B S and Stern A 1980 *Nucl. Phys.* B **164** 429
Brickell F and Clark R S 1974 *J. Diff. Geom.* **9** 557
Cariñena J F and Ibort L A 1983 *J. Phys. A: Math. Gen.* **16** 1
Crampin M 1981 *J. Phys. A: Math. Gen.* **14** 2567
—— 1983a *J. Phys. A: Math. Gen.* **16** 3755
—— 1983b *Phys. Lett.* **95A** 466
Crampin M, Marmo G and Rubano C 1983 *Phys. Lett.* **97A** 88
Currie D I and Saletan E J 1966 *J. Math. Phys.* **7** 967
Darboux G 1894 *Leçons sur la Théorie Générale des Surfaces* III Partie (Paris: Gauthier-Villars)
De Filippo S, Marmo G, Salerno M and Vilasi G 1982 *Phys. Lett.* **117B** 418
—— 1983 *Lett. Nuovo Cimento* **37** 105
—— 1984 *Nuovo Cimento* B **83** 97
Ferrario C, Lo Vecchio G, Marmo G and Morandi G 1985 *Lett. Math. Phys.* **9** 140
Frölicher A and Nijenhuis A 1956 *Indag. Math.* **23** 338
Giandolfi F, Marmo G and Rubano C 1981 *Nuovo Cimento* B **66** 34
Grifone J 1972 *Ann. Inst. Fourier* **22** 287, **22** 291
Havas P 1957 *Nuovo Cimento Suppl.* **5** 363
Helmholtz H 1887 *J. Reine Angew. Math.* **100** 137
Henneaux M 1981 *Hadronic J.* **4** 2137
—— 1982 *J. Phys. A: Math. Gen.* **15** L93
—— 1984 *J. Phys. A: Math. Gen.* **17** 75
Hojman S and Harleston H 1981 *J. Math. Phys.* **22** 1414
Klein J 1983 *Proc. IUTAM-ISIMM Symposium (Torino, June 1982) Atti Acc. Sci. Torino* **1** 177
Lax P 1968 *Commun. Pure Appl. Math.* **21** 467
—— 1975 *Commun. Pure Appl. Math.* **28** 141
—— 1976 *SIAM Rev.* **18** 351
Lutzki M 1982 *J. Phys. A: Math. Gen.* **15** L87
Marmo G 1982 *Proc. Meeting on Geometry and Physics (Florence, October 1982)*
Marmo G and Rubano C 1983 *Nuovo Cimento* B **78** 70
—— 1985 *On the Uniqueness of the Lagrangian Description for Charged Particles in External Fields, Preprint* University of Naples
—— 1986 *Phys. Lett.* **119A** 321
Marmo G and Saletan E J 1978 *Hadronic J.* **1** 955
Marmo G, Saletan E J, Simoni A and Vitale B 1985 *Dynamical Systems* (New York: Wiley)
Sarlet W 1983 *J. Phys. A: Math. Gen.* **16** L229
Sarlet W and Cantrijn F 1978 *Hadronic J.* **1** 101
Thirring W 1978 *A Course in Mathematical Physics I: Classical Dynamical Systems* (Berlin: Springer)